Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(02)00078-3
J. Appl. Maths Mechs, Vol. 66, No. 4, pp. 601–604, 2002

0021–8928/02/$—see front matter

A PROBLEM OF THE BENDING OF A PLATE FOR A DOUBLY-CONNECTED DOMAIN BOUNDED BY POLYGONS†

G. A. KAPANADZE

Tbilisi

e-mail: svanadze@gol.ge

(*Received* 12 *July* 2001)

The problem of the bending of an isotropic elastic plate, bounded by two convex polygons is considered. It is assumed that the internal boundary of the plate is simply supported and normal bending moments act on each section of the external contour in such a way that the angle of rotation of the middle surface of the plate is a piecewise-constant function. With respect to the complex potentials, which express the bendings of the middle surface (Goursat's formula), the problem is reduced to a Riemann–Hilbert boundary-value problem for a circular ring, the solution of which is constructed in analytic form. Estimates are given of the behaviour of these potentials in the neighbourhood of the corner points.

1. FORMULATION OF THE PROBLEM

Suppose S is a doubly-connected domain in the z plane of the complex variable, bounded by convex polygons A and B. We will assume that A is the external boundary of the domain S, and B is the internal boundary, and we will denote by A_j $(j = 1, \ldots, q)$ and B_j $(j = 1, \ldots, p)$ the vertices (and their affixes), and by $L_0^{(k)}$ and $L_1^{(k)}$ the sides of the polygons A and B. The values of the internal angles of the domain S at the vertices A_j and B_j will be denoted by $\pi\alpha_j^0$ $\pi\beta_j^0$, while the angles between the x axis and outward normals to the contours L_1 and L_0 will be denoted by $\beta(t)$ and $\alpha(t)$. The positive direction on $L = L_0 \cup L_1$ $(L_0 = \cup L_0^{(k)}, L_1 = \cup L_1^{(k)})$ will be assumed to be that which keeps domain S to the left.

We will assume that a rigid plank is riveted to each section of the boundary L_0 and the plate is bent by normal moments applied to the planks in such a way that the angles of rotation of the middle surface of the plate take piecewise-constant values, while the contour L_1 is simply supported.

We will consider the following problem: it is required to obtain the bending deflection $w(x, u)$ of the middle surface of the plate if the values of the principal bending moment M_n are known on each section of the boundary contour L_0.

2. SOME ADDITIONAL PROPOSITIONS

Dirichlet's problem for a circular ring. Suppose D $(1 < |z| < R)$ is a circular ring bounded by circles $l_0(|z| = R)$ and $l_1(|z| = 1)$. We will consider the following problem: it is required to obtain a function $\varphi(z) = u + iv$, holomorphic in the ring D, with respect to the boundary condition

$$\operatorname{Re}[\varphi(t)] = f_j(t), \quad t \in l_j, \quad j = 0, 1 \tag{2.1}$$

The necessary and sufficient condition for problem (2.1) to be solvable has the form

$$\int_0^{2\pi} f_0(t)d\vartheta = \int_0^{2\pi} f_1(t)d\vartheta \tag{2.2}$$

while the solution itself is given by the formula

$$\varphi(z) = \frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{l_0}\frac{f_0(t)}{t - R^{2j}z}dt + \int_{l_1}\frac{f_1(t)}{t - R^{2j}z}dt\right] + ic_1 + c_2, \quad c_2 = \frac{1}{4\pi}\int_0^{2\pi} f_1(t)d\vartheta \tag{2.3}$$

where c_1 is an arbitrary real constant.

†*Prikl. Mat. Mekh.* Vol. 66, No. 4, pp. 616–620, 2002.

The conformal mapping of a doubly-connected domain, bounded by polygons, onto a circular ring. Suppose S is the doubly-connected domain considered above. We will consider the following problem: it is required to find the form of the function $z = \omega(\zeta)$ which conformally maps the circular ring $D(1 < |\zeta| < R)$ onto the domain S.

The derivative of the function $\omega(\zeta)$ is the solution of the Riemann–Hilbert problem for the circular ring D [1].

$$\begin{aligned}&\operatorname{Re}[i\,te^{-i\alpha(t)}\,\omega'(t)] = 0, \quad t \in l_0(|\zeta| = R)\\ &\operatorname{Re}[i\,te^{-i\beta(t)}\,\omega'(t)] = 0, \quad t \in l_1(|\zeta| = 1)\end{aligned} \tag{2.4}$$

The necessary condition for this problem to be solvable in the class $h(b_1, \ldots, b_p)$ (regarding this class see [2]) has the form

$$\prod_{k=1}^{q} R^2 a_k^{\alpha_k^0 - 1} \prod_{j=1}^{p} b_j^{\beta_j^0 - 1} = 1 \tag{2.5}$$

(a_k and b_k are the inverse images of the points A_k and B_k), while the solution of this class itself is given by the formula

$$\begin{aligned}\omega'(\zeta) = k^0 \prod_{k=1}^{q} \left(1 - \frac{\zeta}{a_k}\right)^{\alpha_k^0 - 1} \prod_{k=1}^{q} b_k^{(1-\beta_k^0)/2} \left(1 - \frac{b_k}{\zeta}\right)^{\beta_k^0 - 1} \times \\ \times \prod_{j=1}^{\infty} \prod_{k=1}^{q} \left(1 - \frac{\zeta}{R^{2j}\alpha_k}\right)^{\alpha_k^0 - 1} \left(1 - \frac{a_k}{R^{2j}\zeta}\right)^{\alpha_k^0 - 1} \prod_{k=1}^{p} \left(1 - \frac{\zeta}{R^{2j}b_k}\right)^{\beta_k^0 - 1} \left(1 - \frac{b_k}{R^{2j}\zeta}\right)^{\beta_k^0 - 1}\end{aligned} \tag{2.6}$$

where k^0 is an arbitrary real constant.

3. SOLUTION OF THE PROBLEM

According to the approximate theory of the bending of a plate, the bending deflection $w(x, y)$ of the central surface in the case considered satisfies the equation

$$\Delta^2 w(x, y) = 0, \quad z = x + iy \in S \tag{3.1}$$

and the boundary conditions

$$M_n(t) = f(t); \ \partial w / \partial s = 0, \ t \in L_0; \ w(t) = 0, \ M_n(t) = 0, \ t \in L_1 \tag{3.2}$$

Using well-known formulae [2–4] we have

$$\begin{aligned}&w(x, y) = \operatorname{Re}[\bar{z}\varphi(z) + \chi_0(z)], \quad z \in S\\ &\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = e^{-iv(t)}[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}]\\ &2D_0(\sigma - 1)d[\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}] = \left[M_n(t) + i\int_0^s N(t)ds + ic\right]dt\\ &\psi(z) = \chi_0'(z), \ v(t) = a(t), \ t \in L_0, \ v(t) = \beta(t), \ t \in L_1; \ \varkappa = (\sigma + 3)(1 - \sigma)^{-1}\end{aligned} \tag{3.3}$$

where D_0 is the cylindrical stiffness of the plate, σ is Poisson's ratio, N is the shearing force and c is a real constant.

By virtue of condition (3.2) and formula (3.3) with respect to the required functions $\varphi(z)$ and $\varphi(z)$ we obtain the problem

$$\begin{aligned}&\operatorname{Re}[ie^{-iv(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})] = 0\\ &\operatorname{Re}[ie^{-iv(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)})] = F_j^*, \quad t \in L_j, \ j = 0, 1\end{aligned} \tag{3.4}$$

where

$$F_1^*(t) = c^{(1)}(t),\ \ t \in L_1;\ \ F_0^*(t) = [2D_0(\sigma - 1)]^{-1} \int_0^s M_n(t)ds + c^{(0)}(t),\ \ t \in L_0$$

$$c^{(1)}(t) = c_k^{(1)} = \text{const},\ \ t \in L_k^{(1)},\ \ c^{(0)}(t) = c_k^{(0)} = \text{const},\ \ t \in L_0^{(k)}$$

The constants $c_k^{(j)}$ ($j = 0, 1$) are unknown in advance and must be determined when solving the problem in such a way that the functions $\varphi(z)$ and $\bar{z}\varphi'(z) + \psi(z)$ extend continuously into the domain $S + L$.

Boundary conditions (3.4) can be written in the form

$$\begin{aligned} &\operatorname{Re}[ie^{-i\nu(t)}\varphi(t)] = F_j(t) \\ &\operatorname{Re}[ie^{-i\nu(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})] = 0,\ \ t \in L_j,\ \ j = 0, 1 \end{aligned} \tag{3.5}$$

where $F_j(t) = [\varkappa + 1]^{-1} F_j^*(t), j = 0, 1$.

Suppose the function $z = \omega(\zeta)$ conformally maps the circular ring $D(1 < |\zeta| < R)$ onto the domain S. We will assume that the contour $l_0(|\zeta| = R)$ transfers into L_0, while the contour $l_1(|\zeta| = 1)$ transfers into L_1. We will denote the inverse images of the points A_k and B_k by a_k and b_k.

Putting $\varphi(z) = \varphi[\omega(\zeta)] = \varphi_0(\zeta)$, from boundary conditions (3.5) in terms of the function $\chi(\zeta) = \zeta^{-1}\varphi_0(\zeta)$ we obtain the Riemann–Hilbet boundary-value problem for a circular ring D

$$\operatorname{Re}[i\sigma e^{-i\nu(\sigma)}\chi(\sigma)] = \Psi_j(\sigma),\ \ \sigma \in L_j,\ \ j = 0, 1;\ \ \Psi_j(\sigma) = F_j[\omega(\sigma)] \tag{3.6}$$

Consider the homogeneous problem corresponding to problem (3.6) when $\Psi_j(\sigma) \equiv 0$. It has the form of problem (2.4), and, hence, the solution of this homogeneous problem of the class $h(b_1, \ldots, b_p)$ has the form $\chi^0(\zeta) = \omega'(\zeta)$, where the function $\omega'(\zeta)$ is defined by formula (2.6).

Hence, we can represent the function $e^{2i\nu(\sigma)}$ in the form

$$e^{2i\nu(\sigma)} = \sigma\omega'(\sigma)\ [\overline{\sigma\omega'(\sigma)}]^{-1},\ \ \sigma \in l_j,\ \ j = 0, 1$$

Boundary conditions (3.6) can now be written in the form

$$\Omega(\sigma) - \overline{\Omega(\sigma)} = \Theta_j(\sigma),\ \ \sigma \in l_j,\ \ j = 0, 1 \tag{3.7}$$

where $\Omega(\zeta) = \varphi_0(\zeta)[\zeta\omega'(\zeta)]^{-1}$, $\Theta_j(\sigma) = -2ie^{i\nu(\sigma)}[\sigma\omega'(\sigma)]^{-1}\Psi_j(\sigma), j = 0, 1$

The necessary and sufficient condition for problem (3.7) to be solvable has the form

$$\int_0^{2\pi} \Theta_1(e^{i\vartheta})d\vartheta = \int_0^{2\pi} \Theta_2(\mathrm{Re}^{i\vartheta})d\vartheta \tag{3.8}$$

and the solution itself is given by the formula

$$\Omega(\zeta) = \frac{1}{2\pi i} \sum_{j=0}^{1} K(\zeta; t)\Theta_j(t)dt + c_0^{**},\ \ K(\zeta; t) = \sum_{k=-\infty}^{\infty} \frac{1}{t - R^{2k}\zeta} \tag{3.9}$$

where c_0^{**} is a real constant.

Hence, the solution of problem (3.5) has the form

$$\varphi_0(\zeta) = \Omega(\zeta)\zeta\omega'(\zeta) = -\frac{\zeta\omega'(\zeta)}{\pi(\varkappa + 1)} \sum_{k=-\infty}^{\infty} I_k(\zeta) + c_0^* \tag{3.10}$$

$$I_k(\zeta) = \int_{l_1} \frac{e^{i\beta(\sigma)}c^{(1)}(\sigma)}{(\sigma - R^{2k}\zeta)\sigma\omega'(\sigma)} d\sigma + \int_{l_0} \frac{e^{i\alpha(\sigma)}[f_0(\sigma) + c^{(0)}(\sigma)]}{(\sigma - R^{2k}\zeta)\sigma\omega'(\sigma)} d\sigma$$

$$f_0(\sigma) = [2D_0(\sigma - 1)]^{-1} \int_0^s M_n(t)ds,\ \ c_0^* = -c_0^{**}\pi(\varkappa + 1)$$

Condition (3.6) can be written in the form

$$\int_{l_1} \frac{e^{i\beta(\sigma)}c^{(1)}(\sigma)}{\sigma^2\omega'(\sigma)}d\sigma + \int_{l_0} \frac{e^{i\alpha(\sigma)}[f_0(\sigma)+c^{(0)}(\sigma)]}{\sigma^2\omega'(\sigma)}d\sigma = 0$$

Since the function $\omega'(\zeta)$ at the points a_k has singularities of the form $|\zeta - a_k|^{\alpha_k^0-1}$, for the function $\varphi_0(\zeta)$ to be continuously extendable into the domain $D + 1$ it is necessary and sufficient for the following conditions to be satisfied

$$\sum_{k=-\infty}^{\infty} I_k(a_j) + c_0^* = 0, \quad j = 1, \dots, q \tag{3.11}$$

Using well-known results [2, Section 26] with respect to the behaviour of a Cauchy type integral in the neighbourhood of points of discontinuity of the density, it can be proved that when conditions (3.11) are satisfied the function $\omega'(z) = \varphi_0'(\zeta)[\omega'(\zeta)]^{-1}$ in the region of the points B_k ($k = 1, \dots, p$) satisfies the condition

$$|\varphi'(z)| < M|z - B_k|^{-1/\beta_k^0}, \quad k = 1, \dots, p, \quad M = \text{const}$$

and in the region of the points A_k ($k = 1, \dots, q$) it is bounded. Similarly for the function $\varphi''(z)$ the following limits hold

$$|\varphi''(z)| < M|z - B_k|^{-1-1/\beta_k^0}, \quad k = 1, \dots, p$$

$$|\varphi''(z)| < M|z - A_k|^{1/\alpha_k^0-2}, \quad k = 1, \dots, q$$

We will now find the function $\psi(z)$. To do this we will use conditions (3.5), which we will write in the form [5]

$$\text{Re}[ie^{i\nu(t)}(\psi(t) + P(t)\varphi'(t)] = \Gamma_j(t), \quad t \in L_j, \quad j = 0, 1 \tag{3.12}$$

where

$$\Gamma_j(t) = \text{Re}[ie^{i\nu(t)}(\bar{t} - P(t)\varphi'(t)], \quad t \in L_j, \quad j = 0, 1$$

and $P(t)$ is an interpolation polynomial which satisfies the condition $P(B_k) = \bar{B}_k$ ($k = 1, \dots, p$) and $\bar{B}_k$ is a number conjugate to B_k.

Since the functions $\Gamma_j(t)$, $t \in L_j (j = 0, 1)$ are bounded, the problem of finding the function $\psi(z)$ reduces to the problem investigated above. The solution of problem (3.12) can be constructed in the same way as before, while the conditions for this problem to be solvable (the requirement that the function $\psi(z) + P(z)\varphi'(z)$ should be continuously extendable) will have a form similar to conditions (2.10) and (2.11). All these conditions are represented as an inhomogeneous system with real coefficients with respect to the constants $c_1^*, c_k^{(1)}$ ($k = 1, \dots, p$) and $c_0^*, c_k^{(0)}$ ($k = 1, \dots, q$) (c_1^* is a real constant which occurs when solving problem (2.12)). It has been proved that the above system has a unique solution, and, hence, the problem in question is uniquely solvable.

REFERENCES

1. KAPANADZE, G., Conformal mapping of doubly-connected domain bounded by broken lines on circular ring. *Bull. Georgian Academy Sci.*, 1999, **160**, 3, 435–437.
2. MUSKHELISHVILI, N. I., *Singular Integral Equations: Boundary Problems of Function Theory and Their Applications to Mathematical Physics*. Dover, New York, 1992.
3. LEKHNITSKII, S. G., Some problems related to the theory of the bending of thin plates. *Prikl. Mat. Mekh.*, 1938, **2**, 2, 181–210.
4. FRIDMAN, M. M., Some problems of the theory of the bending of thin isotropic plates. *Prikl. Mat. Mekh.*, 1941, **5**, 1, 93–102.
5. BANTSURI, R. D., The solution of the third fundamental problem of the theory of elasticity for doubly-connected domains bounded by broken lines. *Dokl. Akad. Nauk SSSR*, 1978, **243**, 4, 882–885.

Translated by R.C.G.